

Individual and Multiagent Four - Dimensional Assets Portfolio Selection

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Abstract

A mathematical theory in order to extract additional information from historical assets prices (Internal Rate of Return or IRR, ranking, and risk) is provided, which is used to ascertain how well test subjects acting as decision-makers decide upon the best portfolio they can choose, working individually or in groups when compared to the “optimal” portfolio obtained using a zero-one integer programming model. Both quantitative (price, IRR and risk) and qualitative (ranking) information are provided by the interface, allowing the test subjects to decide which of the four dimensions (asset price, IRR, ranking or risk) are plotted, two at a time. The interface has been specifically designed to allow decision makers to make strategic considerations.

Keywords: Finance, Engineering Economic Analysis, Assets, Strategy, Portfolio, Selection, Prices, Internal Rate of Return (IRR), Ranking, Risk

Introduction

The purpose of this paper is to present theory and mathematical models that allow individual or groups of decision makers to decide upon a set of assets (corporate stocks or government bonds) as to which of them to include in a given portfolio. In order to test how well the decision makers do, a zero-one integer programming model is proposed in order to obtain the “optimal” portfolio.

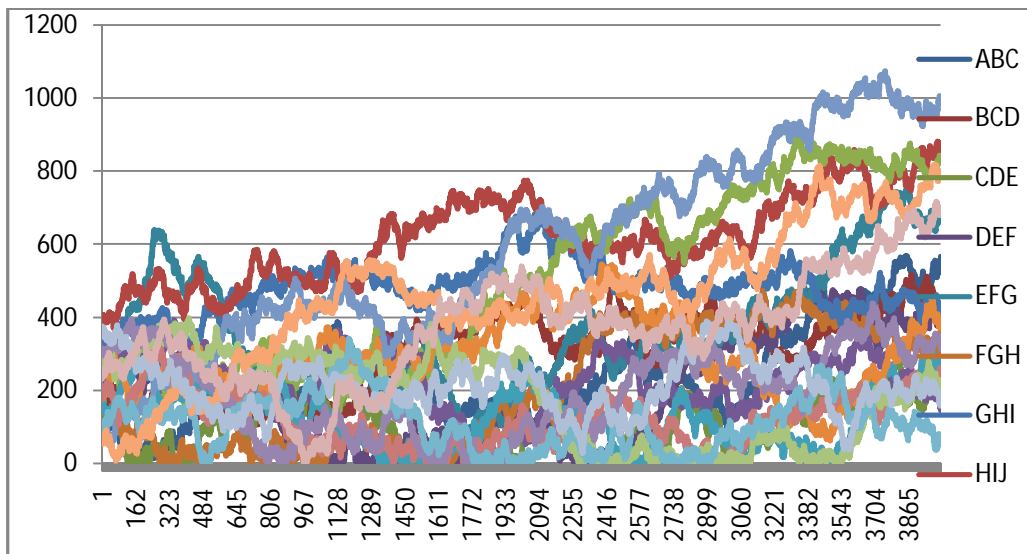
Clearly, in practice there are strategic considerations that decision makers usually do when deciding upon their portfolio.

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Thus, the “optimal” is not necessarily the best in reality, but test subjects will be given the same explanation so that they decide merely on the information provided by the interface and hopefully not on their “gut feelings”. That is the reason why the information is presented using not only quantitative but also qualitative information. An easy-to-use graphic user interface is created so that test subjects can interact with the system when taking their decisions and making strategic considerations.

Obtaining data from reality is expensive². Consequently, data was simulated consisting of historical assets prices from 1990 to 2000 (a total of 4018 days; $n = 4018$). Based on those historical prices and the theory and mathematical models here discussed, a total of four data dimensions are generated: asset prices (source data), Internal Rate of Return (IRR), ranking also called priority, and risk. These are the four dimensions which are presented to the user as the user sees fit. The set of twenty assets prices are plotted in Figure 1 so that the reader can see they truly behave somewhat like real historical assets prices.

Figure 1: Historical Asset Prices for a Set of Twenty Different Simulated Asset Prices



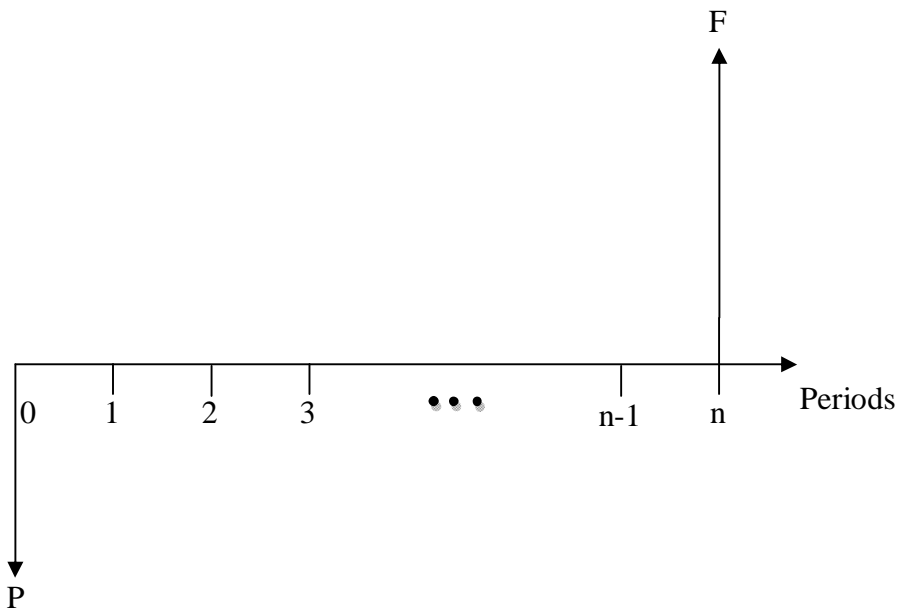
² For more information, refer to the websites of investment groups or stock exchanges.

Knowledge from several areas of expertise is combined in this paper in order to make sense of the information and to create a good decision-making framework. Engineering Economic is probably one of the most important areas used. Also, there is Linear Programming and, in particular, Zero-One Integer Linear Programming. Teamwork is another area, as well as Fuzzy Logic and Heuristics, combined with Human-Computer Interaction theory. Clearly, Finance is always present.

Theory

To understand the formula used to obtain annualized daily IRR values for asset k (IRR_{kj}) based on historical asset prices, it is necessary to start with some very basic concepts of Engineering Economic Analysis (Newman, Eschenbach & Lavelle, 2004). Assume there are n periods between the first inflow of money (such as lending), which occurs at time 0, and a final payment, which occurs at time n . Figure 2 describes the situation.

Figure 2: Initial Lend and Final Payment at Period n



The equation that relates P and F assuming an interest rate i per period is shown in equation (1).

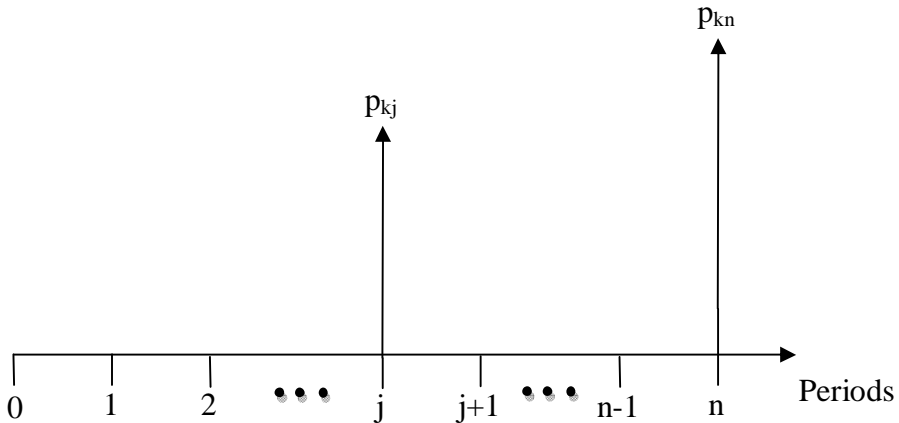
$$F = P(1 + i)^n \quad (1)$$

If the lend is received at period j and paid at period n , then the relationship changes to the one described by equation (2), where $j \leq n$.

$$F = P(1 + i)^{n-j} \tag{2}$$

However, when dealing with asset prices for a given asset k , the values of the prices are given, and the interest rate, now called the IRR is what is of interest. Let p_{kj} and p_{kn} be the prices of asset k at periods j and n , respectively, where $j \leq n$. Figure 3 illustrates the new situation.

Figure 3: Asset k prices at period's j and n



The equations now become interesting. The relationship between p_{kj} and p_{kn} is given by equation (3), where p_{kj} and p_{kn} are the daily prices of asset k at days j and n , respectively, and IRR_{kj} is the Internal Rate of Return between days j and n .

$$p_{k_n} = p_{k_j} \left(1 + IRR_{kj} \right)^{n-j} \tag{3}$$

Solving for $1 + IRR_{kj}$ yields equation (4), where s is the total number of assets or maximum portfolio size.

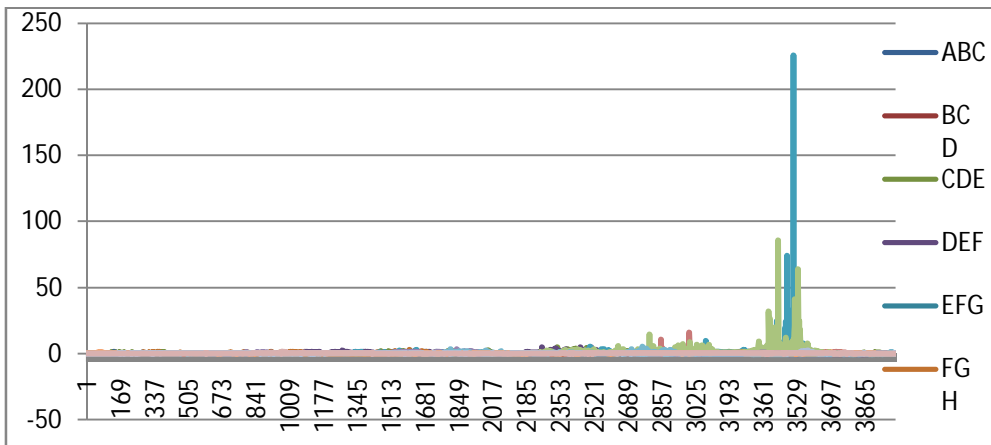
$$\left(\frac{p_{k_n}}{p_{k_j}} \right)^{\frac{1}{n-j}} = 1 + IRR_{kj} \forall k = 1, \dots, s \text{ and } j = 1, \dots, n; j < n \tag{4}$$

The yearly IRR_{kj} for asset k at day j is given by equation (5).

$$IRR_{kj} = \begin{cases} \left(\left(\frac{p_{kn}}{p_{kj}} \right)^{\frac{365}{n-j}} - 1 \right) \times 100\% & \text{where } j \text{ is in a nonleap year} \\ \left(\left(\frac{p_{kn}}{p_{kj}} \right)^{\frac{366}{n-j}} - 1 \right) \times 100\% & \text{where } j \text{ is in a leap year} \\ \left(\frac{p_{kn}}{p_{kj}} - 1 \right) \times 100\% & \forall \text{ where } j \text{ is in the last year} \end{cases} \quad (5)$$

Notice that now IRR_{kj} is the yearly Internal Rate of Return for asset k at day j . The bottom term in equation (5) is not raised to any power because otherwise the results obtained make no sense, since in that case $n-j$ would be less than 365 or 366. Figure 4 shows the historical IRR_{kj} for all twenty assets from the example used. Notice that it is possible to have negative values for IRR_{kj} . Also, realize that there can be a considerable fluctuation in the values for IRR_{kj} .

Figure 4: IRR_{kj} Values for all Twenty Assets and $n=4018$ Days



The average IRR_k for asset k is calculated according to equation (6), where n is the total number of days in the block of prices and IRR data ($n = 4018$ in the case of the example here used).

$$IRR_k = \frac{\sum_{j=1}^n IRR_{kj}}{n} \quad \forall k = 1, \dots, s \quad (6)$$

The average price for asset k (p_k) can be calculated in a similar way according to equation (7), where n is also the same value as in equation (6), that is, the total number of days (4018 in the example).

$$p_k = \frac{\sum_{j=1}^n p_{kj}}{n} \quad \forall k = 1, \dots, s \quad (7)$$

Now that the first two dimensions have been calculated for each asset k (p_k and IRR_k), it is time to calculate the ranking or priority of each asset. For that, a fuzzy logic estimator is used. Zadeh (1965, 1997) started the work on fuzzy logic. Fuzzy logic was developed as an important tool for systems control and complex industrial processes, as well as home and entertainment electronics, diagnostic systems and others. Contrary to conventional logic, where there are only two options: true (1) or false (0), in fuzzy logic there is a range of alternatives numerically represented as a number between 0 and 1. For example, consider the case of tall or short people. In conventional logic it might be argued that tall people are higher and 1.80 meters and short people are shorter than 1.80, so that someone who is 1.81 meters would be considered tall and someone who is 1.79 meters would be considered short. However, there is only a difference of merely 2 centimeters between the two. In fuzzy logic, however, a maximum stature is specified, such as 2 meters, and then the given stature of any individual is divided by 2 meters, which in principle should provide values between 0 and 1. Of course, this is just a simple example, since there could be people taller than 2 meters.

Heuristics (Tversky&Kahneman, 1974; Gigerenzer, 2008; Gilovich, Griffin &Kahnemann, 2002), on the other hand, are mental shortcuts or simple rules of thumb to solve problems, such as "if it is expensive is better", or "if it is fat is because it eats too much". Of course, they might be wrong, but they help people in making quick decisions when the situation is complex. The fuzzy logic estimator known here as ranking is combined with heuristics in the following way: if the ranking is greater than 0 and less than 0.33, then the asset is a low priority asset (color red); if the ranking is greater or equal than 0.33 but less than 0.67 then the asset is medium priority (color yellow); and if the ranking is greater or equal than 0.67 and less or equal than 1 then the asset is high priority (color green).

But how such ranking indicator is created so that it follows the previous logic? First, consider the ranking based on price for asset k (Z_{p_k}). The maximum variation in such case is given as $\text{Max}\{p_i\}-\text{Min}\{p_i\}$, where $i = 1, \dots, s$. In the case of price, the minimum the actual value for the price of asset k the better. Thus, $\text{Max}\{p_i\}-p_k$ is the highest when p_k is the lowest and it is zero when p_k is the highest. In this way, the ranking evaluator for price (Z_{p_k}) is given as indicated in equation (8).

$$Z_{p_k} = \frac{\text{Max}\{p_i\}-p_k}{\text{Max}\{p_i\}-\text{Min}\{p_i\}} \forall i = 1, \dots, s \text{ and } k = 1, \dots, s \quad (8)$$

The case for IRR is somewhat different, since the higher the IRR the better. Now the maximum difference is also given in the same way: $\text{Max}\{\text{IRR}_i\}-\text{Min}\{\text{IRR}_i\}$. Thus, $\text{IRR}_k-\text{Min}\{\text{IRR}_i\}$ is the highest (one) when IRR_k is the highest and it is zero when IRR_k is the lowest. In this way, the ranking evaluator for IRR (Z_{IRR_k}) is given according to equation (9).

$$Z_{\text{IRR}_k} = \frac{\text{IRR}_k-\text{Min}\{\text{IRR}_i\}}{\text{Max}\{\text{IRR}_i\}-\text{Min}\{\text{IRR}_i\}} \forall i = 1, \dots, s \text{ and } k = 1, \dots, s \quad (9)$$

Finally, let w_p and w_{IRR} be the weights assigned to the price and the IRR ranking indicators, respectively, where $w_p + w_{\text{IRR}} = 1$. (In the case of the example used, $w_p = 0.5$ and $w_{\text{IRR}} = 0.5$.) So the final expression for ranking is given according to equation (10).

$$Z_k = (w_p Z_{p_k} + w_{\text{IRR}} Z_{\text{IRR}_k}) \times 100\% \forall k = 1, \dots, s \quad (10)$$

Now there are three dimensions: price, IRR and ranking. The final dimension is risk. But how could risk be calculated based on historical asset prices? It is important to distinguish between risk and uncertainty (Bammer & Smithson, 2008). Risk is the likelihood or probability of failure, whereas uncertainty is the variability of the relevant outcomes for a given risk or eventuality. Brealey and Myers (2007) define risk as the condition that more things might happen (at present) than will happen (in the future). Uncertainty, on the other hand, is the degree to which an identified threat or risk (at present time after prior assessment) will (presumably, based on experience, historical data or assumptions) vary. Uncertainty is an identified (and quantified) risk. Still, the degree to which such identified risk will vary is unknown. Uncertainty thus constitutes the 'known unknowns' because although a specific risk has been identified, its actual impact is still unknown.

Non-identified risks are ‘unknown unknowns’ because, generally speaking, a risk is non-quantified uncertainty about something not yet considered to be possible as a future outcome. It is assumed throughout that risk identification has been successfully and thoroughly carried out and will focus on the risk due to the uncertainty for the most relevant variable here identified, which is price, since the historical assets prices are the basis for all calculations.

Let μ_k be the average of the prices for asset k. Also, let σ_k be the standard deviation of the prices for asset k. According to the central limit theorem (Devore, 2012), assuming the prices behave following a normal distribution³, 95% of all the data will be between $\mu_k - 2\sigma_k$ and $\mu_k + 2\sigma_k$. The average for all the prices of asset k (μ_k) has already been calculated in equation (7) as p_k . What remains to be calculated is the standard deviation of all prices for asset k (σ_k), which is the square root of the variance (σ_k^2). The variance for the prices of asset k is calculated according to equation (11).

$$\sigma_k^2 = \frac{\sum_{j=1}^n (p_j - p_k)^2}{n-1} \quad \forall k = 1, \dots, s \quad (11)$$

The uncertainty for asset k is thus calculated according to equation (12).

$$\Delta_k = 2\sigma_k \quad \forall k = 1, \dots, s \quad (12)$$

Let g_k represent the ratio between the uncertainty of asset k (Δ_k) and the actual average (mean) price of such asset k (p_k), as indicated in equation (13). Then it is possible to normalize such quantity called uncertainty ratio as indicated in equation (14). The latter is called the normalized risk of asset k (nr_k). Clearly, such “normalized risk” is not correct in order to indicate the risk of asset k. The sum of all the nr_k is equal to 1. That is why the risk of asset k (r_k) is further defined according to equation (15) so that the higher normalized risk represents a 100% (1) risk and the lower normalized risk represents a 0% (0) risk. If $\text{Max}\{nr_i\} - \text{Min}\{nr_i\} = 0$, then $r_k = 100\%$. Suppose for the example that the decision-makers risk tolerance, K , is 66.67%.

³ All the historical assets prices have been checked using 7-bar histograms. Some of the assets prices are very close to a normal distribution, others not that much so, but the assumption remains.

$$g_k = \frac{\Delta_k}{p_k} \forall k = 1, \dots, s \quad (13)$$

$$nr_k = \frac{g_k}{\sum_{i=1}^s g_i} \forall k = 1, \dots, s \quad (14)$$

$$r_k = \frac{nr_k - \text{Min}\{nr_i\}}{\text{Max}\{nr_i\} - \text{Min}\{nr_i\}} \times 100\% \forall i = 1, \dots, s \text{ and } k = 1, \dots, s \quad (15)$$

In this way, all four dimensions of interest are defined (asset price, IRR, ranking and risk).

Zero-One Integer Programming Model

There has to be a way to make a comparison between what the test subjects choose as their assets portfolio and some standard "optimal solution". Such model is proposed in this section. But before proposing the model, let stand back and review what project selection is.

Project selection is one of the first and most critical activities in project management. Deciding from a pool of available and competing projects which ones should be undertaken (thus assigning limited resources to them) and which ones should not be undertaken or terminated is a complex decision. Overall value maximization, balance among dimensions, and business strategy should be considered. The very essence of portfolio management portrayed by Cooper, Edgett&Kleinschmidt(2007) as a "dynamic decision process... constantly up-dated and revised... [where] new projects are evaluated, selected and prioritized; existing projects may be accelerated, killed or de-prioritized; and resources are allocated or re-allocated to the active projects" increases the difficulty. Furthermore, portfolio selection is a process characterized by uncertainty and changing information: new opportunities arise, multiple goals as well as strategic considerations are required, interdependence among projects (either when competing for scarce resources or when synergies are achieved) exists, not to mention multiple decision-makers and locations. Consequently, a mathematical model seems to be the best long term approach to tackle such a complex decision making process.

According to Meredith and Mantel (1995) project selection methods can be classified as nonnumeric (qualitative) or numeric (quantitative). The sacred cow, operating necessity, competitive necessity, product line extension, and the comparative benefit model are among the qualitative methods.

Profitability models (payback period, average rate of return, NPV, IRR, profitability index, as well as others that subdivide the elements of the cash flow, include terms of risk or uncertainty or consider the effect on other projects on the organization) and scoring models (weighted and non-weighted zero-one factor models with or without constraints usually solved using integer programming as well as goal programming when multiple objectives are given) are among the quantitative methods.

A decision support system for project portfolio selection is presented by Archer and Ghasenzadeh(1998). The asset selection model proposed is a maximization zero-one integer programming scoring model that is more extensive because it explicitly considers risk as well as ranking. For an alternative zero-one integer programming model, refer to work by Ghasemzadeh, Archer & Iyogun (1999).

However, the present paper, although related to project portfolio selection, it is on assets portfolio selection. That makes the zero-one integer programming model here proposed similar, but also different. Let the column vector $\mathbf{x} = [x_1, x_2, \dots, x_s]$ represent whether or not asset k is selected (a one if it is selected, a zero if it is not). Also, let the row vector $\mathbf{i} = [IRR_1, IRR_2, \dots, IRR_s]$ be the IRR of asset k . Let the row vector $\mathbf{p} = [p_1, p_2, \dots, p_s]$ be the average asset price vector for all assets $k = 1, \dots, s$. The average asset price for all assets, P , is given according to equation (16).

$$P = \frac{\sum_{i=1}^s p_i}{s} \quad (16)$$

Let the row vector $\mathbf{z} = [Z_1, Z_2, \dots, Z_s]$ be the ranking of asset $k = 1, \dots, s$. For the example here discussed, the minimum ranking tolerated is $Z = 0.5$ (or 50%). Let the row risk vector $\mathbf{r} = [r_1, r_2, \dots, r_s]$ be defined as the risk of asset $k = 1, \dots, s$, where the maximum risk tolerated as defined before is $K = 66.67\%$. Finally, let the set P_j be the set of all pre-required assets j for asset i ⁴ and E_j be the set of all mutually exclusive assets for assets i and j ⁵.

⁴ That is, if asset j is included, asset i must be selected, if not, asset i may or may not be selected. That is, asset i is a pre-required asset for asset j . The set P_j may contain several different assets i .

⁵ That is, both assets i and j cannot be selected at the same time, either i is selected or j is selected, but not both or, alternative, none of them is selected. The set E_j may contain several assets i , but if j is mutually exclusive with i , i is also mutually exclusive with j .

MAXIMIZE

$$\mathbf{ix} \quad (17)$$

SUBJECT TO:

$$\text{a) price constraint:} \quad (\mathbf{p}-P)\mathbf{x} \leq 0 \quad (18)$$

$$\text{b) ranking constraint:} \quad (\mathbf{z}-Z)\mathbf{x} \geq 0 \quad (19)$$

$$\text{c) risk constraint:} \quad (\mathbf{r}-K)\mathbf{x} \leq 0 \quad (20)$$

$$\text{d) pre-required constraint:} \quad x_i - x_j \geq 0 \quad \forall i \in P_j \quad (21)$$

$$\text{e) mutually exclusive constraint:} \quad x_i + x_j \leq 1 \quad \forall i \in E_j \quad (22)$$

$$\text{f) technical constraint:} \quad x_k = \begin{cases} 1 \\ 0 \end{cases} \quad \forall k = 1, \dots, s \quad (23)$$

The first equation for the zero-one integer programming model is the objective function. That is simply to maximize the sum of the IRR of all the assets selected, which is shown in equation (17).

The next equation is for the average price of all the assets selected to be less or equal than the average price chosen, P , which is shown in equation (18a). With some algebraic manipulation equation (18b) is obtained. By moving the term to the right of equation (18b) to the left, results in equation (18c). Equation (18c) is written in vector notation as shown in equation (18).

$$\frac{\sum_{k=1}^s x_k p_k}{\sum_{k=1}^s x_k} \leq P \quad (18a)$$

$$\sum_{k=1}^s x_k p_k \leq \sum_{k=1}^s P x_k \quad (18b)$$

$$\sum_{k=1}^s (p_k - P) x_k \leq 0 \quad (18c)$$

The case for the ranking equation is different. In this case, the average ranking of all the assets selected has to be greater or equal than the target ranking, $Z = 0.5$ (50%) in the example. Equation (19a) shows this idea expressed mathematically. Rearranging terms and expressing them in vector notation results in equation (19).

$$\frac{\sum_{k=1}^s x_k Z_k}{\sum_{k=1}^s x_k} \geq Z \quad (19a)$$

For risk, there is equation (20a) expressing that the average risk has to be less or equal to the decision-makers risk tolerance, K . Expressing it in vector format results in equation (20).

$$\frac{\sum_{k=1}^S x_k r_k}{\sum_{k=1}^S x_k} \leq K \quad (20a)$$

The set of equations from (17) to (23) define the zero-one integer programming model proposed:

Group Decision Making Considerations

When facing the need to reach a consensus in a given decision setting, it is difficult for decision makers with different levels of expertise or interest to reach a consensus when there is also uncertainty (Palomares&Martinez, 2014).

One of the most important advantages of group decision making over individual decision making is creativity, that is, the opportunity group decision making offers individuals to openly discuss and challenge each other in order to reach, presumably, a better decision (Entani, 2013).

One of the important issues in group decision making is reaching consensus versus majority rule. Although consensus is difficult to reach, sometimes some form of consensus or soft consensus can be reached. Methodologies have been developed for studying reaching some form of consensus or at least managing such consensus (Zhang & Dong, 2013).

Nevertheless, it seems the rule of the majority instead of trying to reach consensus seems more effective to apply in group decision making. However, the number of individuals in the group in order to reach majority rule is important. Apparently, having three individuals in the group makes it difficult to reach majority rule. Up to seven individuals in the group seem to be a better number of individuals to include in group decision making (Taylor, Hewitt, Reeves, Hobbs & Lawless, 2013).

When considering a group of undergraduate students for reaching a consensus decision, it is very important to consider the communication style of the ones that mostly influence the final decision. Apparently, persuasive communication in the form of rational persuasion is most effective (Ali, Ho-Abdullah & Mastor, 2012).

It seems that majority rule leads to faster decisions and less argumentation among participants than consensus rule. Groups of up to five individuals have been considered for this purpose (McKoy et al., 2012).

Another important aspect to consider when reaching group decisions is learning style. Having groups with different learning styles helps members to foster the learning styles in which they are weak (Ofiaz & Turunc, 2012). Assigning blame in group decisions is also important. Apparently, when a team suffers a loss, blame occurs more in the presence of successful complementary peers than when a successful substitute exists (Zultan, Gerstenberg & Lagnado, 2012).

From the brief literature review conducted, it seems more efficient to ask the group of undergraduate students to decide which assets to include in the portfolio by following majority rule rather than consensus rule, although if they can reach consensus when deciding would be better if they find that reaching such consensus motivated constructive debate among them. The maximum number of individuals to include in group decisions is also an issue. According to the literature reviewed, such maximum number could be five or seven. Due to physical limitations and a matter of practicality, it has been decided to have a maximum of four members in a decision making team, because the maximum recommendation is up to five members according to McCoy et al. (2012). However, experiments with single individuals making the decision as well as groups of two individuals will also be included in this experimental setting. Comparing the average accuracy of the results as well as the average amount of time used to reach decisions is an important feature of this work.

Human-Computer Interaction

Visualization includes the study of both image synthesis and understanding, including several academic disciplines, fields of study, and multiple domains of interest. Lohse, Biolsi, Walter & Rueter (1994) state that the need for classification schemes is based on the fact that classification is at the heart of each and every scientific field.

Classifications “structure systematic domains of interest and provide concepts to develop theories towards identifying anomalies and predict future research needs”.

Graphics and images can be characterized either as functional (focused on the intended use and the purpose of the graphic material) or structural (focused on the form of the image rather than its content). Graphics encode information quantitative using position and magnitude of geometric objects. The numerical data in one, two or three dimensions are plotted in a Cartesian or polar coordinate system. Common graphs include coordinate systems of scatterplots, categorical, linear, piled bar, bar, pie, boxes, fan, surface response, histograms, stars, polar coordinates and face Chernoff charts.

Preece et al. (1996) identify several techniques to represent numerical data: scatterplots, line or curved charts, area, band, strata or surface charts, bar, columns or histogram charts, pie charts, simulated meters and star, circular or pattern charts.

Bertin (1983) defines understanding as “simplifying, reducing a vast amount of «data» to a small number of categories of «information» that we are capable of taking into account when dealing with a certain problem.” Preece et al. (1996) discuss what is known in the literature of the Human-Computer Interaction (HCI) as the magic number 7 ± 2 , related to the short term memory, which shows that human beings are capable of remembering between 5 and 9 figures at the same time. This is one of the reasons why a good HCI display is critical, since it allows users to consider several numbers at the same time if the display presents information in a significant way. Although this concept of understanding seems very precise, the human brain is in fact much more capable than what it seems to be assumed. The mind is capable of making abstractions, synthetize several elements from reality and put them together using not only short term memory, but also long term memory.

The design proposed is a polysemy graphic system, in which the meaning of the individual signs is deduced and results from the collection of signs. For the purposes of this paper, perception deals with the ability of any individual (or group of) expert(s) to find relationships between the images and the real world, in an attempt to reach the best assets portfolio. The information displayed on the screen or printed in a sheet of paper is the result of summarizing in a chart, based on mathematical models, the combination of all the available data from historical records (keeping a small database) and the entries obtained from test subjects during each session.

There are four dimensions of data: Internal Rate of Return (IRR), price, ranking or priority and risk. Four dimensions of data have to be displayed in a significant way. The problem is what Bertin (1981) calls the insuperable barrier: up to three kinds of numerical data can be constructed in a single image, producing a scatterplot, in which the objects of the third dimension are typically denoted as the third dimension or "z" axis, allowing movement around the three-dimensional coordinate system. But four dimensions of data have to be displayed (and not three). Is there anything that can be done to avoid not showing the general relations of the complete set of data?

Considering what can be represented in a flat piece of paper or flat computer screen, a graphic system can include eight variables plus the two or three axis of the two or three dimensional space: a) size, b) value, c) texture, d) color, e) orientation, and f) shape. Cleveland & McGill (1984) order from more to least exact the ten elementary perceptual tasks:

- 1) Position on a common scale.
- 2) Position on non-aligned scales.
- 3) Longitude, direction and angle.
- 4) Area.
- 5) Volume and curvature.
- 6) Shading or color saturation.

The problem is how to display four dimensions. Cleveland (1993) explores the use of scatterplots in a multi-panel display of four rows and four columns for hypervariate data.

Each pair of variables is plotted on a scatterplot on each panel; alternatively, it is possible to have the dimensions on the x and y axis in combo-boxes and allow the user to choose which of the two out of the four dimensions to display at the same time.

Experimental Setting

One of the basic goals of this paper is to present two alternative mathematical models. On one hand is the mathematical model that calculated IRR, ranking and risk for each asset based on historical assets prices.

On the other hand is the zero-one integer programming optimization model that, based on the information calculated with the initial mathematical model (price, IRR, ranking and risk for each asset) and considering maximum average price, minimum ranking and maximum risk while maximizing overall IRR, yields the “optimal” portfolio. The word “optimal” is between apostrophes because real decision-makers would consider not only the raw information but also strategic considerations that may lead them to include in the portfolio an asset that would otherwise not be included.

A total of one hundred individuals are to participate in the testing in order to compare the performance of individuals or groups of individuals (groups of two, three or four) when deciding which assets to include in their portfolio while interacting with a specially designed Graphic User Interface (GUI) for such purpose. The GUI allows them to select assets while not allowing assets with pre-required assets not selected to be selected or having two mutually exclusive assets selected at the same time. Also, the GUI shows the users the value of the total IRR, the average assets price that must not exceed the average asset price given, the average ranking that needs to be at least higher than 50% and the average risk that needs not be higher than 66.67%. The GUI also records the time it takes to reach their final decision. All the information including participant name, age, study program and semester is recorded for data analysis purposes.

The first ten experiments have a single test subject. The second group of ten experiments has two test subjects each. The third group of ten experiments has three test subjects each. Finally, the fourth group of ten experiments has four test subjects each. Notice that the total number of test subjects is $10+2(10)+3(10)+4(10) = 100$ test subjects (and a total of 40 experiments).

Let S^t be the score of experiment or test t , where $t = 1, \dots, 40$. Also, let T^t be the time (in seconds) it took the test subject(s) of test t to reach their final decision on the assets portfolio. The score S^t is calculated according to equation (24), where x_k^t is whether or not asset k in test t was selected and x_k^* is the “optimal” result for asset k as to whether or not it must be selected (these numbers are 1 if selected and 0 if not selected), and s is the total number of assets. Notice that the score S^t is a value between 0 and 1, when 0 is worst score and 1 is perfect score.

$$S^t = \frac{s - \sum_{k=1}^s (x_k^t - x_k^*)^2}{s}, \text{ where } t = 1, \dots, 40 \quad (24)$$

Besides storing the score and the time for each test, the string with the portfolio chosen with s binary numbers (either 1 for asset k selected or 0 for asset k not selected) is also stored.

It is also important to consider the times in order to have a combined measure of both the score and the relative time it takes to reach each decision. Having T^t be the time it took for the test number t , where $t = 1, \dots, 40$, and since the lower the T^t value is the better, a time score TS^t is calculated according to equation (25), where $t = 1, \dots, 40$.

$$TS^t = \frac{\text{Max}\{T^j\} - T^t}{\text{Max}\{T^j\} - \text{Min}\{T^j\}}, \text{ where } j = 1, \dots, 40 \text{ and } t = 1, \dots, 40 \quad (25)$$

Then, the final score, FS^t , where $t = 1, \dots, 40$, is given according to equation (26).

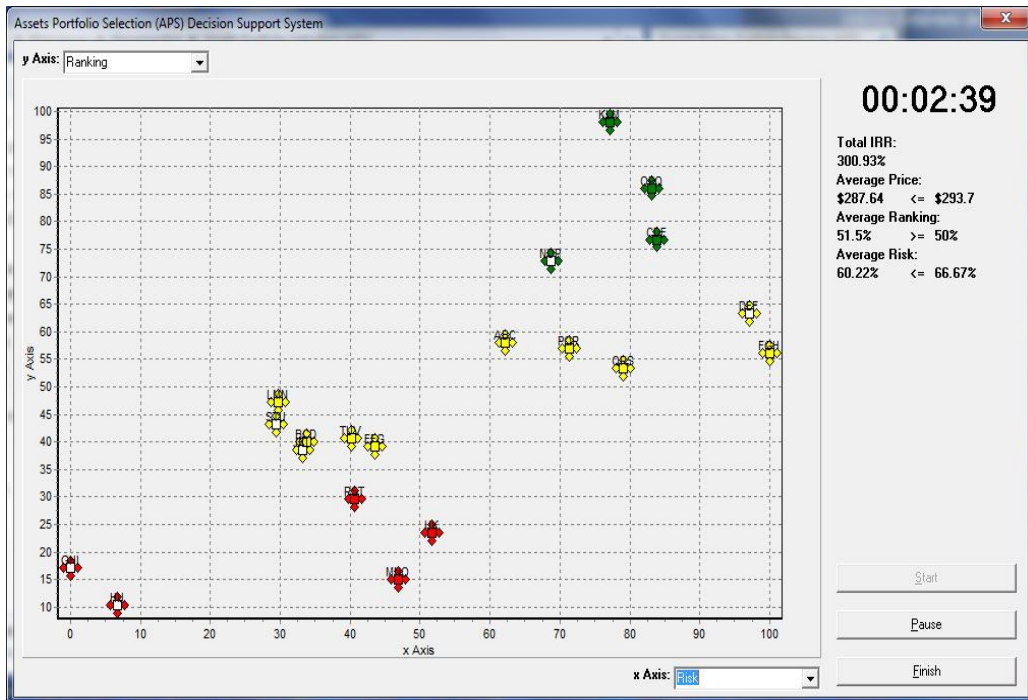
$$FS^t = S^t TS^t, \text{ where } t = 1, \dots, 40 \quad (26)$$

Since there are 10 experiments for each type of decision (individual: $t = 1, \dots, 10$; groups of two: $t = 11, \dots, 20$; groups of three: $t = 21, \dots, 30$; and groups of four: $t = 31, \dots, 40$, it makes a total of 40 experiments). The average FS^t for each type of experiment has to be calculated, as well as the minimum value, maximum value, standard deviation, median, mode, and whether or not the behavior of the variables in the set of 40 experiments is normal, using a range of 3 intervals for the histogram and in order to calculate the mode (Abdous&Theodorescu, 1998).

Discussion and Conclusion

An example of the GUI used by test subjects with the "optimal" portfolio selected is shown in Figure 5.

Figure 5: Example of the GUI Used with Test Subjects having the “Optimal” Portfolio Selected



Descriptive statistics for the relevant variables are shown in Table 1.

Table 1: Descriptive Statistics for the Relevant Variables of the 100 Test Subjects

Criteria	Age	Semester	S ^t	T ^t	TS ^t	FS ^t
Minimum	17	1	0.4000	103	0.0000	0.0000
Maximum	45	10	1.0000	1,877	1.0000	0.8322
Mean (□)	21.44	5.05	0.7975	671.59	0.6795	0.5377
Standard deviation (□)	3.2077	2.7391	0.1285	254.9279	0.1437	0.1413
Median	21	5	0.8500	673.5	0.6784	0.5318
Mode	17	5.5	0.7000	990	0.5000	0.4161
Normal?	No	Yes	Yes	Yes	Yes	Yes

Table 2 shows the significant correlations (higher than the absolute value of 0.10) between the eight variables of interest (t, team members, age, semester, S^t, T^t, TS^t and FS^t), where t = 1,...,40. The correlations are based on Pearson's formula (Devore, 2012). Notice that the list of all 100 test participating subjects is used.

Table 2: Significate Correlations (Pearson's r) between the eight Variables of Interest (t , team Members, age, and Semester, S^t , T^t , TS^t and FS^t)

Variables	t	Team members	Age	Semester	S^t	T^t	TS^t
Team members	<i>0.9111</i>	-					
Age	-	-	-				
Semester	<i>-0.1215</i>	-	<i>0.5953</i>	-			
S^t	<i>0.2953</i>	0.3442	-0.1460	-0.1556	-		
T^t	<i>-0.1228</i>	-	-	-	0.1151	-	
TS^t	<i>0.1228</i>	-	-	-	-0.1151	<i>-1.0000</i>	-
FS^t	<i>0.3321</i>	0.3385	-	-	<i>0.5829</i>	<i>-0.7283</i>	<i>0.7283</i>

Let first explain and get rid of the obvious and unimportant correlations, which are marked in italics in Table 2. Clearly, the high correlation between the experiment number (t) and the number of team members is due to the fact that there is the same number of team members for $t = 1, \dots, 10$ (1 individual), $t = 11, \dots, 20$ (2 members), $t = 21, \dots, 30$ (3 members) and $t = 31, \dots, 40$ (4 members). The correlations between test number ($t = 1, \dots, 40$) and the remaining variables is also not worth considering.

The correlation between age and semester is also obvious. Generally speaking, the higher the age, the higher the semester the individual should be in. Nevertheless, it is worth mentioning that such correlation should be higher, which means that either some individuals enter the undergraduate in computer science or computer engineering at a higher age or that there is a considerable rate of course's failures in the population, which changes the relationship between age and semester.

Also, the correlation between the score obtained at test t (S^t , $t = 1, \dots, 40$) and the final score obtained at test t (FS^t) should not be considered because the latter is the result of multiplying the score by the standardized time score (TS^t), which is reflected in equation (26).

Notice how the correlation between the final score (FS^t , $t = 1, \dots, 40$) and the time score (T^t , $t = 1, \dots, 40$) and the standardized time score (ST^t , $t = 1, \dots, 40$) is also meaningless because, directly or indirectly, they are based on equations (25) and (26).

Finally, there is a perfect and negative correlation between the time of test t ($T^t, t = 1, \dots, 40$) and the standardized time score ($TS^t, t = 1, \dots, 40$). This perfect and negative correlation is meaningless, since TS^t is completely based on T^t by applying equation (25).

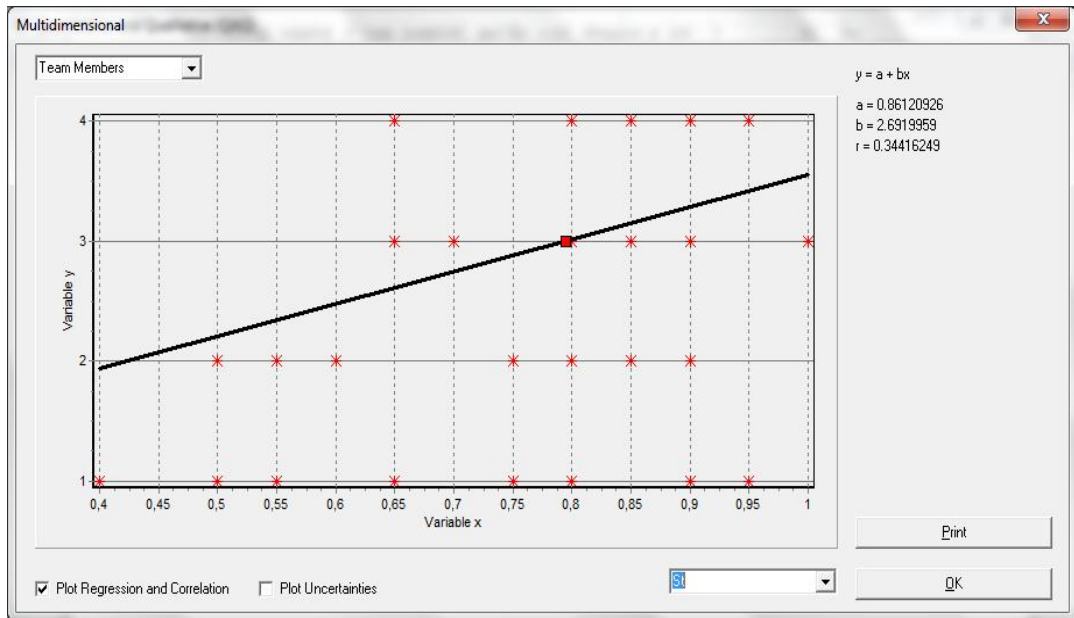
Let now focus on the important correlations. The first and most important of all is the correlation between the number of team members and the score obtained at test t ($S^t, t = 1, \dots, 40$). This correlation ($r = 0.3442$), which is backed by the correlation between the number of test subjects and the final score ($FS^t, t = 1, \dots, 40$), $r = 0.3385$, indicates that there is a positive effect between the number of test subjects and the score ($S^t, t = 1, \dots, 40$) or final score ($FS^t, t = 1, \dots, 40$) obtained. That is, the higher the number of members in the team, the better they do at solving the problem. This means that more people certainly think better than one person. The correlation and regression as well as the scatterplot resulting between team members and S^t is shown in Figure 6.

Also, the slightly negative but nevertheless somewhat significant correlation between age and score ($S^t, t = 1, \dots, 40$), which is $r = -0.1460$, indicates that the older the test subject the worse they do. The same applies for the correlation between semester and score ($r = -0.1556$). Apparently, the above is due to the fact that younger people tend to have a wider point of view because apparently they do not take things for granted. Also, people deeper in their undergraduate career (that is, with a higher semester) show the same behavior.

The slightly positive but nonetheless significant correlation between score ($S^t, t = 1, \dots, 40$) and time ($T^t, t = 1, \dots, 40$) means that the more time the individuals or team take to make their decision, the better they do, which makes sense. The same but negative correlation between score ($S^t, t = 1, \dots, 40$) and standardized time score ($TS^t, t = 1, \dots, 40$) indicates the same pattern. Remember that due to equation (25) the standardized time score ($TS^t, t = 1, \dots, 40$) is perfectly inverted with respect to time ($T^t, t = 1, \dots, 40$).

It was observed that the test subject when working in groups holding the mouse was given more importance when reaching a decision.

Figure 6: Regression and Correlation with Scatter Plot between Team Members (Y Variable) and Score Obtained (X Variable)



The conclusions are clear. First, the higher the number of individuals making a decision (up to four team members), the better the decision they reach. Second, the younger the test subjects are the better they decide, which apparently means that expertise is not a plus, probably due to the fact that more experienced subjects relied on such expertise and not simply on the raw data provided by the interface. Third, the more time the individuals take to reach their decision, the better they do.

Further research may consider having more team members reaching decisions or having the team members scattered physically, but nonetheless with some form of communication between them. Also, it is important to further assess the real usefulness of the interface when making strategic consideration, given the result obtained that more experienced subjects (of a higher age) tend to do worse when making their decision.

References

- Abdous, B. & Theodorescu, R. (1998). "Mean, median, mode IV", *StatisticaNeerlandica*, 52(3): 356-359.
- Ali, Z., Ho-Abdullah, I. & Mastor, K.A. (2012). "Undergraduate communication style in consensual small groups decision making", *Procedia – Social and Behavioral Sciences*, 59: 612-617.
- Archer, N.P. & Ghasemzadeh, F. (1998). "A decision support system for project portfolio selection", *International Journal of Technology Management*, 16(1/2/3): 105-114.
- Bammer, G. & Smithson, M. (2008). *Uncertainty and Risk: Multidisciplinary Perspectives*. Sterling, VA: Earthscan.
- Bertin, J. (1981). *Graphics and Graphic Information Processing*. Berlin: Walter de Gruyter.
- Bertin, J. (1983). *Semiology of Graphics: Diagrams, Networks, Maps*. London: The University of Wisconsin Press.
- Brealey, R. & Myers, S. (2007). *Principles of Corporate Finance*. New York, NY: McGraw-Hill.
- Cleveland, W.S. & McGill, R.. (1984). "Graphical Perception: Theory, Experimentation, and Application to the Development of Graphical Methods", *Journal of the American Statistical Association*, 79(387): 531-554.
- Cleveland, W.S. (1993). *Visualizing Data*, Murray Hill, NJ: AT&T Bell Laboratories.
- Cooper, R.G., Edgett, S.J. & Kleinschmidt, E.J. (2007). "Portfolio Management for New Product Development – Results of an Industry Practices Study", *Product Development Institute*, www.stage-gate.com.
- Devore, J. (2012). *Probability and Statistics for Engineering and the Sciences*. Boston, MA: Brooks/Cole.
- Entani, T. (2013). "Encouragement of Group Decisions Beyond the Sum of the Individuals Based on Possible Estimations", *Procedia Computer Science*, 22: 846-854.
- Ghasemzadeh, F., Archer, N., & Iyogun, P. (1999). "A zero-one model for project portfolio selection and scheduling", *Journal of the Operational Research Society*, 50: 745-755.
- Gigerenzer, G. (2008). "Why Heuristics Work", *Perspectives on Psychological Science*, 3(1): 20-29.
- Gilovich, T., Griffin, D. & Kahnemann, D. (2002). *Heuristics and Biases: The Psychology of Intuitive Judgment*. New York, NY: Cambridge University Press.
- Lohse, G.L., Biolsi, K., Walter, N. & Rueter, N.H. (1994). "A Classification of Visual Representations", in *Communications of the ACM*, 37(12): 36-49.
- McKoy, M., Spitler, S., Zuchegno, K., Enslein, A., Hobbs, S., Reeves, R., Patton, T. & Lawless, W.F. (2012). "An Experimental Physiological Approach of Group Decision Making: Consensus Rule vs Majority Rule", *Procedia Technology*, 5: 475-480.
- Meredith, J.R. & Mantel, S.J., Jr. (1995). *Project Management: A Managerial Approach*, New York: John Wiley & Sons.
- Newnan, D.G., Eschenbach, T.G. & Lavelle, J.P. (2004). *Engineering Economic Analysis*. New York, NY: Oxford University Press.
- Oflaz, M. & Turunc, T. (2012). "The effect of learning styles on group work activities", *Procedia – Social and Behavioral Sciences*, 46: 1333-1338.

- Palomares, I. & Martínez, L. (2014). "Low-dimensional visualization of experts' preferences in urgent group decision making under uncertainty", *Procedia Computer Science*, 29: 2090-2101.
- Preece, J., Rogers, Y., Sharp, H., Benion, D., Holland, S. & Carey, T. (1996). *Human-Computer Interaction*, Essex, England: Pearson Education Limited.
- Taylor, E., Hewitt, K., Reeves, R.A., Hobbs, S.H. & Lawless, W.F. (2013). "Group decision making: Consensus rule versus majority rule", *Procedia Technology*, 9: 498-504.
- Tversky, A. & Kahneman, D. (1974). "Judgment under Uncertainty: Heuristics and Biases", *Science*, 185(4157): 1124-1131.
- Zadeh, L.A. (1965). "Fuzzy Sets", *Information and Control*, 8(3): 338-353.
- Zadeh, L.A. (1997). "Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic", *Fuzzy Sets and Systems*, 90: 111-127.
- Zhang, B. & Dong, Y. (2013). "Consensus rules with minimum adjustments for multiple group decision making", *Procedia Computer Science*, 17: 473-481.
- Zultan, R., Gerstenberg, T. & Lagnado, D.A. (2012). "Finding fault: Causality and counterfactuals in group attributions", *Cognition*, 125: 429-440.